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COMPACTNESS THEOREMS FOR COUPLED YANG-MILLS FIELDS

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Abstract

In this paper, we consider Coupled Yang-Mills fields on vector bundle E over compact Riemannian manifold M. Under appropriate conditions on the curvature and the Higgs field, two compactness theorems are proved.

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1 Introduction

Let M be a n-dimensional manifold with a Riemannian metric g, and E a vector bundle over M with a compact Lie group G as its structure group. A connection A of E can be given by specifying a covariant derivative

$$D_A : C^\infty(E) \to C^\infty(E \otimes \Omega^1 M).$$

In local trivialization of E, D_A is of the form $d + \alpha$ for some Lie(G)-valued 1-form α. The curvature of A is a Lie(G)-valued 2-form F_A, which is equal to D_A^2. As usual, it measures deviation from the symmetry of second derivatives. Suppose that the bundle E has a Riemannian structure. All connections here are required to be compatible with the Riemannian structure.

The Determinant of the volume bundle over M is a line bundle of conformal weight n. We denote by L, the determinant bundle raised to the $\frac{1}{n}$ power. Sections of this bundle are constant in a fixed coordinate system but have weight 1 under scale transformations. Let the Higgs field Φ be a section of $E \otimes L$, and the mass m be a section of L. With these definitions, the Yang-Mills-Higgs functional is defined in the following

$$YMH(A, \Phi) = \int_M [|F_A|^2 + |\nabla_A \Phi|^2 + \frac{\lambda}{4}(|\Phi|^2 - m^2)^2] \tag{1.1}$$

where λ is a constant. The Yang-Mills-Higgs equations are

$$\left\{ \begin{array}{l} D_A^* F_A = -D_A \Phi \otimes \Phi^*, \\ D_A^* D_A \Phi = -\frac{\lambda}{2}(|\Phi|^2 - m^2)\Phi \end{array} \right. \tag{1.2}$$

which are the Euler-Lagrange equations of the Yang-Mills-Higgs functional and generalize the Yang-Mills equations. One pair (A, Φ) satisfying the Yang-Mills-Higgs equations will be called by one Coupled Yang-Mills fields.

Gauge transformation σ is a section of $Aut(E)$ which acts on connections, curvature forms, and Higgs fields according to the transformations:

$$\begin{array}{ll} \sigma^*(A) & = \sigma^{-1} d\sigma + \sigma^{-1} A \sigma, \\ F_{\sigma^* A} & = \sigma^{-1} F_A \sigma, \\ \sigma^*(\Phi) & = \sigma \circ \Phi. \end{array} \tag{1.3}$$

The pair (A, Φ) is gauge equivalent to $(\overline{A}, \overline{\Phi})$ if there is a gauge transformation σ such that $\overline{A} = \sigma^*(A)$ and $\overline{\Phi} = \sigma^*(\Phi)$. It is obvious that the Yang-Mills-Higgs equations are invariant under gauge transformations.

In analytical aspect of the Yang-Mills theory, two most fundamental results are the K. Uhlenbeck's compactness theorem and removable singularity theorem ([Uh1],[Uh2]). Later, the removable singularity theorem of K.Uhlenbeck was extended to coupled Yang-Mills fields by T.H.Parker, L.M.Sibner and R.J.Sibner, P.D.Smith, T.H.Otway ([Pa1], [SS], [Si], [Sm], [OS], [O]).

The modulo space of Yang-Mills connections (or coupled Yang-Mills fields) is the quotient of the set of solutions of the Yang-Mills equation (Yang-Mills-Higgs equations with fixed λ and m) by the gauge group, which consists of all gauge transformations. It is well-known that this modulo space may not be compact. Given any sequence of Yang-Mills connections $\{A_i\}$ with a uniformly bounded L^2-norm of curvature, Uhlenbeck ([Uh1]) (also see [Na], [Ti]) proved that by taking a subsequence if necessary, A_i converges to, modulo gauge transformations, a Yang-Mills connection A in the smooth topology outside a closed subset $S_b(\{A_i\})$ of Hausdorff codimension at least 4. If M is a 4-dimensional compact manifold, the blow-up locus consists of finitely many points, and the limiting connection A can be extended to be a Yang-Mills connection on the whole manifold with smaller L^2-norm of curvature ([Uh1], [Uh2]). With M of higher dimension, G.Tian [Ti] studied the geometric structures of the blow-up loci of Yang-Mills connections. He also proved a removable singularity theorem for stationary Yang-Mills connections. Particularly, this implies that the limiting connection A extends to be become a smooth connection on $M \setminus S$ for a closed subset S with vanishing $(n-4)$-dimensional Hausdorff measure $H^{n-4}(S) = 0$.

In this paper we consider the compactness property of sequences of coupled Yang-Mills fields (A_i, Φ_i) with a uniformly bounded Yang-Mills-Higgs energy $YMH(A_i, \phi_i)$. We first derive a monotonicity formula of Coupled Yang-Mills fields. Using R.Schoen and K.Uhlenbeck's argument in [Sc] for harmonic maps (the method has been extended to pure Yang-Mills fields by H.Nakajima [Na]), we can deduce a ϵ estimates for Coupled Yang-Mills fields. Then, discussing like that in [Ti] (for pure Yang-Mills fields case), we can obtain the following compactness theorem.

Theorem 1.1 *Let (A_i, Φ_i) be a sequence of smooth Coupled Yang-Mills fields over M with fixed $\lambda > 0$ and m; then there is a subsequence (A_i, Φ_i) and gauge transformations σ_k, such that $\sigma_k^*(A_k, \Phi_k)$ converges to a Coupled Yang-Mills fields (A, Φ) outside a closed subset S of Hausdorff codimension at least 4.*

In the second part of this paper, we want to deduce another compactness theorem under different assumption. Noting that the Yang-Mills-Higgs energy is not conformally ("scale") invariant unless the dimension of underground Riemannian manifolds are four. In higher dimension, only $L^{\frac{n}{4}}$ norm of the Yang-Mills-Higgs energy density is conformally invariant. Under a uniform bound on $L^{\frac{n}{4}}$ norm of the Yang-Mills-Higgs energy density, we deduce the following theorem.

Theorem 1.2 *Let E be a vector bundle over compact Riemannian manifold M of dimension $n > 4$, (A_i, Φ_i) be a sequence of smooth Coupled Yang-Mills fields on E with fixed $\lambda > 0$ and m. Assume that*

$$\int_M [|F_{A_i}|^2 + |\nabla_{A_i}\Phi_i|^2 + \frac{\lambda}{4}(|\Phi_i|^2 - m^2)^2]^{\frac{n}{4}} \leq C_0, \tag{1.4}$$

where C_0 is a positive constant, then there is a subsequence (A_α, Φ_α) converge to, modulo gauge transformations, a smooth Coupled Yang-Mills field (A, Φ) in C^∞-topology on M.

The ideal of the proof of theorem 1.2 is similar as that in [Zh] for pure Yang-Mills connection case. First, We will use the local curvature estimate and the conformally invariant of $L^{\frac{n}{4}}$ to deduce that there exists a subsequence (A_k, Φ_k) (modulo gauge transformations) converge to a Coupled Yang-Mills connection (A_∞, Φ_∞) in smooth topology outside a blow up set $\tilde{S}$ which is at most finite points. Secondly, using the removable singularity theorem due to L.M.Sibner[Si], we know that the limiting field can be extends to a smooth one on M. Furthermore, discussing as that in [Ti], we will construct non-trivial bubbling fields on R^m as (A_k, Φ_k) approach to (A_∞, Φ_∞) if the blow up set is non-empty. On the other hand, when $n > 4$, we can prove an non-existence theorem for coupled Yang-Mills fields which will show that bubbling fields, in fact, are not exist, then the blow up set $\tilde{S}$ must be empty. So the subsequence (A_k, Φ_k) (modulo gauge transformations) converge to a smooth Coupled Yang-Mills field in C^∞ topology on M.

2 Preliminary Results

Let $\pi : E \to M$ is a vector bundle over a Riemannian manifold (M, g) with a compact Lie group G as its structure group. A connection A on E is defined by specifying a covariant derivative

$$D = D_A : C^\infty(E) \to C^\infty(E \otimes \Omega^1 M).$$

where $C^\infty(E)$ denotes the space of C^∞ sections of the bundle E. In a local trivialization $(U_\alpha, \varphi_\alpha)$ of E, the covariant derivative takes the form

$$D = d + A_\alpha, A_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathbf{g}$$

Denote the Lie algebra of G by $\mathbf{g}$ and the adjoint and automorphism bundles by AdE and $AutE$. Assume also E has a metric compatible with the action of G and an inclusion $G \subset SO(r)$. We use the metric on G induced by the trace inner product metric on $SO(r)$.

The gauge group is $\mathbf{G} = C^\infty(AutE)$ which consists of all smooth sections of the bundle $AutE$. Any σ in $\mathbf{G}$ is called a gauge transformation. Two smooth connections A_1 and A_2 of E are equivalent if there is a gauge transformation σ such that $A_2 = \sigma(A_1)$, where $\sigma(A)$ be the connection with $D_{\sigma(A)} = \sigma \cdot D_A \cdot \sigma^{-1}$. One can easily show $\sigma(A) = \sigma \cdot A \cdot \sigma^{-1} - d\sigma \cdot \sigma^{-1}$.

For any connection A of E, its curvature F_A measures the extent to which covariant derivative fail to commute. Then $F_A = D_A^2$ is a section of $\wedge^2 T^*M \otimes AdE$. In each local trivialization $(U_\alpha, \varphi_\alpha)$, we have

$$F_A = dA_\alpha + A_\alpha \wedge A_\alpha. \tag{2.1}$$

The Determinant of the volume bundle over M is a line bundle of conformal weight n. We denote by L, the determinant bundle raised to the $\frac{1}{n}$ power. Sections of this bundle are constant in a fixed coordinate system but have weight 1 under scale transformations. The Higgs field Φ is a section of $E \otimes L$. The mass m is defined to be a section of L, and hence, constant in a fixed coordinate system, but having weight 1 under scale changes. (For a careful and rigorous discussion of conformal weights, see Parker [Pa1], [Pa2]). From these, we construct the Yang-Mills-Higgs Energy with these definitions, the Yang-Mills-Higgs functional is defined in the following

$$YMH(A,\Phi) = \int_M [|F_A|^2 + |\nabla_A \Phi|^2 + \frac{\lambda}{4}(|\Phi|^2 - m^2)^2] dv_g. \tag{2.2}$$

where λ is a constant (in this paper, we only consider the case $\lambda > 0$), and the norms are taken with respect to Riemannian metric g and fixed invariant inner products on the bundles AdE, L and E.

Coupled Yang-Mills fields (A, Φ) are the critical points for the above Yang-Mills-Higgs functional. Equivalently, the pairs (A, Φ) satisfy the following equations.

$$\begin{cases} D_A^* F_A = -D_A \Phi \otimes \Phi^*, \\ D_A^* D_A \Phi = -\frac{\lambda}{2}(|\Phi|^2 - m^2)\Phi \end{cases} \tag{2.3}$$

where Φ^* is the adjoint of Φ taken with respect to the inner product of E, and D_A^* denotes the adjoint operator of D_A with respect the Killing form of $\mathbf{g}$ and the Riemannian metric g on M. The above equations also be called by Yang-Mills-Higgs equations. It is obvious that the Yang-Mills-Higgs energy is invariant by the gauge transformation. Then, if (A, Φ) is a Coupled Yang-Mills field, so is $\sigma^*(A, \Phi)$ for any gauge transformation σ. In other words, the Yang-Mills-Higgs equations are invariant under the action of the gauge group. From the conformal weights of Higgs fields, one can easily show that the Yang-Mills-Higgs equations are also invariant under re-scaling.

Next, we want to deduce a monotonicity formula of the Yang-Mills-Higgs energy, the discussion is similar as that in [P] (also [Ti]) for pure Yang-Mills connections. Let $\{f_t\}_{|t|<\infty}$ be a one-parameter family of diffeomorphisms of M, and A_0 be a fixed smooth connection of E and D be its associated covariant derivative. Then for any connection A, we can define a family of connections $A^t = f_t^*(A)$ as follows: Denote by τ_t^0 the parallel transport of E associate to A_0 along the path $f_s(x)_{0 \le s \le t}$, here $x \in M$. We define $A^t = f_t^*(A)$ by defining its associated covariant derivative

$$D_X^t v = (\tau_t^0)^{-1}(D_{df_t(X)} \tau_t^0(v))$$

for any $X \in TM$, $v \in \Gamma(E)$; and define $\Phi^t = f_t^* \Phi(x) = (\tau_t^0)^{-1}(\Phi(f_t(x)))$.

Let X be the vector field $\frac{\partial f_t}{\partial t}|_{t=0}$, by direct calculation, we have the following equality.

$$\begin{aligned} \frac{d}{dt} YMH(A^t, \Phi^t)|_{t=0} = \; & - \int_M (|F_A|^2 divX - 4 \sum < F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) >) dv_g \\ & - \int_M (|\nabla_A \Phi|^2 divX - \sum 2 < \nabla_{\nabla_{e_i} X} \Phi, \nabla_{e_i} \Phi >) dv_g \\ & - \frac{\lambda}{4} \int_M (|\Phi|^2 - m^2)^2 divX dv_g \end{aligned} \tag{2.4}$$

Fixed any $p \in M$, let r_p be a positive number with properties: there are normal coordinates $x_1, ..., x_n$ in the geodesic ball $B_p(R_p)$ of (M, g), such that $p = (0, ..., 0)$ and for some positive constant $c(p)$,

$$|g_{ij} - \delta_{ij}| \le c(p)(|x_1|^2 + ... + |x_n|^2),$$
$$|dg_{ij}| \le c(p)\sqrt{|x_1|^2 + ... + |x_n|^2},$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. The constant r_p and $c(p)$ can be chosen depending only on the injective radius at p and the curvature of (M, g). If M is the Euclidean space, we can take $r_p = \infty$ and $c(p) = 0$.

Let $r(x)$ be the distance function from p, and ϕ be a positive function on the unit sphere S^{n-1}. Define vector field

$$X(x) = \xi(r)\phi(\frac{x}{r})r\frac{\partial}{\partial r},$$

where ξ is smooth function with compact support in $B_p(r_p)$. Let $\{e_1, ..., e_n\}$ be any orthonormal basis near p such that $e_1 = \frac{\partial}{\partial r}$. We have

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0,$$
$$\nabla_{\frac{\partial}{\partial r}}X = (\xi' r + \xi)\phi(\theta)\frac{\partial}{\partial r},$$
$$\nabla_{e_i}X = \xi r e_i(\phi)\frac{\partial}{\partial r} + \xi\phi\sum b_{ij}e_j$$

where $|b_{ij} - \delta_{ij}| = O(1)c(p)r^2$, and $i \ge 2$. We will always denote by $O(1)$ a quantity bounded by a constant depending only on n.

Let (A, Φ) be a Coupled Yang-Mills field, applying the above equalities to the first variational formula (2.4), we have

$$\begin{aligned}
&\int_M |F_A|^2(\xi' r + (n-4)\xi)\phi + \int_M |\nabla_A\Phi|^2(\xi' r + (n-2)\xi)\phi + \tfrac{\lambda}{4}\int_M(|\Phi|^2 - m^2)^2(\xi' r + n\xi)\phi \\
&\quad = -\sum\int_M \xi\phi(b_{ij} - \delta_{ij}) < e_j, e_i > (|F_A|^2 + |\nabla_A\Phi|^2 + \tfrac{\lambda}{4}(|\Phi|^2 - m^2)^2) \\
&\qquad +4\sum\int_M \xi\phi(b_{ij} - \delta_{ij}) < F_A(e_j, e_k), F_A(e_i, e_k) > \\
&\qquad +2\sum\int_M \xi\phi(b_{ij} - \delta_{ij}) < \nabla_{e_j}\Phi, \nabla_{e_i}\Phi > \\
&\qquad +4\int_M\{\xi' r\phi|F_A(\tfrac{\partial}{\partial r}, \cdot)|^2 + \xi r < F_A(\tfrac{\partial}{\partial r}, \cdot), F_A(\nabla\phi, \cdot) >\} \\
&\qquad +2\int_M\{\xi' r\phi|\nabla_{\frac{\partial}{\partial r}}\Phi|^2 + \xi r < \nabla_{\frac{\partial}{\partial r}}\Phi, \nabla_{\nabla\phi}\phi >\}
\end{aligned} \tag{2.5}$$

For any τ small enough, we choose $\xi(r) = \xi_\tau(r) = \eta(\frac{r}{\tau})$, where η is smooth and satisfies: $\eta(r) = 1$ for $r \in [0, 1]$, $\eta(r) = 0$ for $r \in [1 + \epsilon, \infty)$, $\epsilon > 0$ and $\eta'(r) \le 0$. Then

$$\tau\frac{\partial}{\partial\tau}(\xi_\tau(r)) = -r\xi'_\tau(r). \tag{2.6}$$

Plugging this into (2.5), and choosing a number a large enough, we obtain:

$$\begin{aligned}
&\tfrac{\partial}{\partial\tau}(\tau^{4-n}e^{a\tau^2}\int_M \xi_\tau\phi|F_A|^2) + \tau^2\tfrac{\partial}{\partial\tau}(\tau^{2-n}e^{a\tau^2}\int_M \xi_\tau\phi|\nabla_A\Phi|^2) \\
&\qquad +\tau^4\tfrac{\partial}{\partial\tau}(\tfrac{\lambda}{4}\tau^{-n}e^{a\tau^2}\int_M \xi_\tau\phi(|\phi|^2 - m^2)^2) \\
&\ge 4\tau^{4-n}e^{a\tau^2}\tfrac{\partial}{\partial\tau}(\int_M \xi_\tau\phi|F_A(\tfrac{\partial}{\partial r}, \cdot)|^2) + 2\tau^{4-n}e^{a\tau^2}\tfrac{\partial}{\partial\tau}(\int_M \xi_\tau\phi|\nabla_{\frac{\partial}{\partial r}}\Phi|^2) \\
&\quad -4\tau^{3-n}e^{a\tau^2}\int_M \xi_\tau r < F_A(\tfrac{\partial}{\partial r}, \cdot), F_A(\nabla\phi, \cdot) > -2\tau^{3-n}e^{a\tau^2}\int_M \xi_\tau r < \nabla_{\frac{\partial}{\partial r}}\Phi, \nabla_{\nabla\phi}\Phi > .
\end{aligned} \tag{2.7}$$

Then, by integrating on τ and letting ϵ tend to zero, we have

$$
\begin{aligned}
&\rho^{4-n}e^{a\rho^2}\int_{B_p(\rho)}\phi(|F_A|^2+|\nabla\Phi|^2+\tfrac{\lambda}{4}(|\Phi|^2-m^2)^2)dv_g \\
&\qquad\qquad -\sigma^{4-n}e^{a\sigma^2}\int_{B_p(\sigma)}\phi(|F_A|^2+|\nabla\Phi|^2+\tfrac{\lambda}{4}(|\Phi|^2-m^2)^2)dv_g \\
&\geq -\int_\sigma^\rho \tau^{3-n}e^{a\tau^2}\int_{B_p(\tau)} r(4|F_A(\tfrac{\partial}{\partial r},\cdot)||F_A(\nabla\phi,\cdot)|+2|\nabla_{\frac{\partial}{\partial r}}\Phi||\nabla_{\nabla\phi}\Phi|)dv_g d\tau \\
&\quad +\int_\sigma^\rho 2\tau^{3-n}e^{a\tau^2}\int_{B_p(\tau)}\phi|\nabla\Phi|^2dv_g d\tau+\int_\sigma^\rho 4\tau^{3-n}e^{a\tau^2}\int_{B_p(\tau)}\tfrac{\lambda}{4}\phi(|\Phi|^2-m^2)^2dv_g d\tau \\
&\quad +\int_{B_p(\rho)\subset B_p(\sigma)} r^{4-n}e^{ar^2}\phi(4|F_A(\tfrac{\partial}{\partial r},\cdot)|^2+2|\nabla_{\frac{\partial}{\partial r}}\Phi|^2)dv_g
\end{aligned}
\tag{2.8}
$$

Taking $\phi\equiv 1$, we obtain the following monotonicity formula for Coupled Yang-Mills fields.

Lemma 2.1(A monotonicity formula): *There exist constants r_P, a depend only on M, such that for any $0<\rho<\gamma<r_P$, we have*

$$
\begin{aligned}
&\rho^{4-n}e^{a\rho^2}\int_{B_p(\rho)}(|F_A|^2+|\nabla\Phi|^2+\tfrac{\lambda}{4}(|\Phi|^2-m^2)^2)dv_g \\
&\qquad\qquad -\sigma^{4-n}e^{a\sigma^2}\int_{B_p(\sigma)}(|F_A|^2+|\nabla\Phi|^2+\tfrac{\lambda}{4}(|\Phi|^2-m^2)^2)dv_g \\
&\geq \int_{B_p(\rho)\subset B_p(\sigma)} r^{4-n}e^{ar^2}(4|F_A(\tfrac{\partial}{\partial r},\cdot)|^2+2|\nabla_{\frac{\partial}{\partial r}}\Phi|^2)dv_g \\
&\quad +\int_\sigma^\rho 2\tau^{3-n}e^{a\tau^2}\int_{B_p(\tau)}|\nabla\Phi|^2dv_g d\tau+\int_\sigma^\rho 4\tau^{3-n}e^{a\tau^2}\int_{B_p(\tau)}\tfrac{\lambda}{4}(|\Phi|^2-m^2)^2dv_g d\tau.
\end{aligned}
\tag{2.9}
$$

Moreover, if $M=R^m$ and g is flat, then the equality holds in (2.9) for $\rho\in(0,\infty)$ and $a=0$.

In the following, we give a ϵ-estimate for Coupled Yang-Mills fields. This estimate was first derived by K.Uhlenbeck [Uh1] (also see [Na], [Ti]) for the curvature of Yang-Mills connections. For convenience, we will denote

$$
h=h(A,\Phi)=\sqrt{|F_A|^2+|\nabla_A\Phi|^2+\frac{\lambda}{4}(|\Phi|^2-m^2)^2}. \tag{2.10}
$$

Theorem 2.2: *Let (A,Φ) be a coupled Yang-Mills field on a G-bundle E over M. Then there are ϵ and constant C_0, which depend only on manifold M, such that for any $p\in M$ and $\rho<r_p$, whenever*

$$
\rho^{4-n}\int_{B_p(\rho)} h^2\,dv_g\leq\epsilon, \tag{2.11}
$$

then

$$
h(p)\leq\frac{C}{\rho^2}(\rho^{4-n}\int_{B_p(\rho)} h^2\,dv_g)^{\frac{1}{2}}. \tag{2.12}
$$

Proof. Using Weitzenbock formula, and Yang-Mills-Higgs equation, we have

$$
\begin{aligned}
\triangle|F_A|^2 &= 2|\nabla_A F_A|^2+2<\nabla_A^2F_A,F_A> \\
&= 2|\nabla_A F_A|^2-2<(D_AD_A^*F_A+R_M\sharp F_A+F_A\sharp F_A),F_A> \\
&\geq 2|\nabla_A F_A|^2-2|F_A|^2(|R_M|+|F_M|)-2|F_A|(|D_A\Phi|^2+|\Phi|^2|F_A|),
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
\triangle|D_A\Phi|^2 = & -2 < (D_A^*D_A + D_AD_A^*)D_A\Phi + R_M\sharp D_A\Phi + F_A\sharp D_A\Phi, D_A\Phi > \\
& +2|\nabla(D_A\Phi)|^2 \\
& = 2|\nabla D_A\Phi|^2 - 2 < R_M\sharp D_A\Phi + F_A\sharp D_A, D_A\Phi > -2 < (D_A^*F_A)\Phi, D_A\Phi > \\
& + < *(*F_A\wedge D_A\Phi), D_A\Phi > +\lambda(|\Phi|^2-m^2)|D_A\Phi|^2 + 2\lambda| < D_A\Phi, \Phi > |^2 \\
& \geq 2|\nabla D_A\Phi|^2 - 2(|R_M|^2 + 4|F_A| + 2|\Phi|^2)|\nabla_A\Phi|^2 \\
& -\lambda||\Phi|^2-m^2||D_A\Phi|^2 + 2\lambda| < D_A\Phi, \Phi > |^2,
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\triangle(|\Phi|^2-m^2)^2 = & 2|\nabla(|\Phi|^2-m^2)|^2 + 4(|\phi|^2-m^2)|D_A\Phi|^2 \\
& -4(|\Phi|^2-m^2) < D_A^*D_A\Phi + F_A\sharp\Phi, \Phi > \\
& \geq 2|\nabla(|\Phi|^2-m^2)|^2 + 2\lambda(|\Phi|^2-m^2)^2|\Phi|^2 \\
& -4||\Phi|^2-m^2||D_A\Phi|^2 - 4||\Phi|^2-m^2||F_A||\Phi|^2,
\end{aligned}
\tag{2.15}
$$

where R_M denote the Riemannian curvature tensor of M. In (), we have used the following equality,

$$
D_A^*D_AD_A\Phi = D_A^*(F_A\Phi) = (D_A^*F_A)\Phi - *(*F_A\wedge D_A\Phi). \tag{2.16}
$$

From above inequalities, we have

$$
\begin{aligned}
\triangle h^2 = & 2|\nabla h|^2 + 2h\triangle h \\
& \geq 2|\nabla F_A|^2 + 2|\nabla_A(D_A\Phi)|^2 + \tfrac{\lambda}{2}|\nabla(|\Phi|^2-m^2)|^2 \\
& -(4m^2 + 8|R_M|)h^2 - (14 + 4\lambda^{-\frac{1}{2}} + 2\lambda^{\frac{1}{2}})h^3.
\end{aligned}
\tag{2.17}
$$

On the other hand, it is easy to check that

$$
|\nabla h|^2 \leq |\nabla|F_A||^2 + |\nabla|D_A\Phi||^2 + \frac{\lambda}{4}|D(|\Phi|^2-m^2)|^2. \tag{2.18}
$$

So we have

$$
\triangle h \geq -(4m^2 + 8|R_M|)h - (14 + 4\lambda^{-\frac{1}{2}} + 2\lambda^{\frac{1}{2}})h^2. \tag{2.19}
$$

Define a function

$$
f(r) = (\rho - 2r)^2 \sup_{x\in B_p(r)} h(x), \tag{2.20}
$$

$r \in [0, \frac{1}{2}\rho]$. Then f is continuous in $[0, \frac{1}{2}\rho]$ with $f(\frac{1}{2}\rho)$, so that f attains its maximum at a certain r_0 in $[0, \frac{1}{2}\rho)$.

First we claim that $f(r_0) \leq 64$ if ϵ is sufficiently small. Assume that $f(r_0) > 64$. Put $b = \sup_{x\in B_p(r_0)} h = h(x_0)$, and taking $\sigma = \frac{1}{4}(\rho - 2r_0)$, then we get

$$
\begin{aligned}
\sup\nolimits_{B_{x_0}(\sigma)} h & \leq \sup\nolimits_{x\in B_p(r_0+\sigma)} h(x) \\
& \leq \tfrac{(\rho-2r_0)^2}{(\rho-2r_0-2\sigma)^2} \sup\nolimits_{x\in B_p(r_0)} h(x) = 4b.
\end{aligned}
\tag{2.21}
$$

Clearly, $\sigma\sqrt{b} \geq 2$. Define a metric $g' = bg$, and denote the Yang-Mills-Higgs energy density with respect to g' by $h_{g'}$, then $h_{g'} = b^{-1}h_g$. Hence,

$$
\sup_{x\in B_2(x_0,g')} h_{g'} \leq 4, \tag{2.22}
$$

where $B_2(x_0, g')$ denotes the geodesic ball of g' with radius 2 and center at x_0.

Since (A, Φ) is a Coupled Yang-Mills field, and the Yang-Mills-Higgs equations are invariant under scaling, from (2.19), we have

$$
\triangle_{g'} h_{g'} \geq -C_1 h_{g'} - C_2 h_{g'}^2, \tag{2.23}
$$

where C_1, C_2 are constants depending only on r_p, m, λ, and the sectional curvature bound of (M,g). Using (2.22), we have

$$\triangle_{g'} h_{g'} \geq -(C_1 + 4C_2) h_{g'}. \tag{2.24}$$

Then, by using either the mean-value theorem or the standard Moser iteration, we have

$$1 = h_{g'}(x_0) \leq C_3 \Big(\int_{B_1(x_0,g')} h_{g'}^2 dv_{g'} \Big)^{\frac{1}{2}}, \tag{2.25}$$

where C_3 is some uniform constant.

On the other hand, by the monotonicity formula (2.9),

$$\begin{aligned} \textstyle\int_{B_1(x_0,g')} h_{g'}^2 dv_{g'} &= (\sqrt{b})^{n-4} \textstyle\int_{B_{x_0}(\frac{1}{\sqrt{b}})} h^2 dv_g \\ &\leq (\tfrac{1}{2}\rho)^{4-n} e^{\frac{a}{4}\rho^2} \textstyle\int_{B_{x_0}(\frac{1}{2}\rho)} h^2 dv_g \\ &\leq 2^{n-4} e^{\frac{a}{4}\rho^2} \epsilon. \end{aligned} \tag{2.26}$$

Combining this with (2.25), we obtain

$$1 \leq C_3 (2^{n-4} e^{\frac{a}{4}\rho^2} \epsilon)^{\frac{1}{2}}. \tag{2.27}$$

But, it is impossible when ϵ is sufficiently small. Then, the claim is proved.

Thus, we have

$$\sup_{x \in B_p(\frac{1}{4}\rho)} \rho^2 h \leq 4 f(r_0) \leq 256.$$

Scaling the metric again, let $\tilde{g} = \rho^{-2} g$, then $h_{\tilde{g}} = \rho^2 h_g$. It follows from the above inequality and (2.24) with g' replaced by g that for some uniform constant C_4,

$$\triangle_{\tilde{g}} h_{\tilde{g}} \geq -C_4 h_{\tilde{g}}. \tag{2.28}$$

Then (2.12) follows from (2.28) and a standard Moser iteration.

□

Using Hölder inequality, we have the following corollary.

Corollary 2.3: *Let (A, Φ) be a Coupled Yang-Mills field on a G-bundle E over M. Then there are ϵ and constant C_5, which depend only on manifold M, such that for any $p \in M$ and $\rho < r_p$, whenever*

$$\int_{B_p(\rho)} h^{\frac{n}{2}} \, dv_g \leq \epsilon, \tag{2.29}$$

then

$$h(p) \leq \frac{C_5}{\rho^2} \Big(\int_{B_p(\rho)} h^{\frac{n}{2}} \, dv_g \Big)^{\frac{2}{n}}. \tag{2.30}$$

Using the monotonicity formula (2.9), one can easily prove (theorem 3.3 [Zh]) the following corollary.

Corollary 2.4: *If (A, Φ) is a Coupled Yang-Mills field over (R^n, g_0) $(n \geq 5)$ and satisfying*

$$\int_{R^n} h(A, \Phi)^{\frac{n}{2}}\, dx < \infty,$$

where g_0 is a flat metric, then $h(A, \Phi) \equiv 0$.

In order to compactify the modulo space of Coupled Yang-Mills fields, we need to use singular Coupled Yang-Mills fields of a certain type. An admissible Coupled Yang-Mills field is a smooth field (A, Φ) defined outside a closed subset S in M, such that (1) $H^{n-4}(S \cap K) < \infty$ for any compact subset $K \subset M$, where $H^{n-4}(\cdot)$ stands for the (n-4)-dimensional Hausdorff measure; (2) (A, Φ) satisfies the Yang-Mills-Higgs equations (2.3) $M \setminus S$; (3) (A, Φ) satisfies $\int_{M \setminus S} h^2\, dV_g < \infty$. Clearly, (A, Φ) is smooth on M if $S = \emptyset$. We will call S the singular set of coupled field (A, Φ).

Furthermore, an admissible Coupled Yang-Mills field (A, Φ) is called stationary if (A, Φ) satisfies

$$\begin{aligned} 0 = \quad & -\int_M (|F_A|^2 div X - 4\sum < F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) >) dv_g \\ & -\int_M (|\nabla_A \Phi|^2 div X - \sum 2 < \nabla_{\nabla_{e_i} X} \Phi, \nabla_{e_i} \Phi >) dv_g \\ & -\frac{\lambda}{4}\int_M (|\Phi|^2 - m^2)^2 div X dv_g \end{aligned} \tag{2.31}$$

for any compactly supported vector field X. Where $\{e_i\}$ is any orthonormal basis of M. If (A, Φ) is a smooth Coupled Yang-Mills field, this follows from the first variation formula (2.4).

Proposition 2.5: *Let $n = \dim M > 4$ and S be a discrete set in M. If (A, Φ) is a Coupled Yang-Mills field on $M \setminus S$ and satisfies $\int_K h^{\frac{m}{2}}\, dV_g < \infty$ for each compact set $K \subset M$; then (A, Φ) is stationary and the monotonicity formula (2.9) still hold on M.*

Proof: Denote

$$\begin{aligned} \Psi(X) = \quad & -\int_M (|F_A|^2 div X - 4\sum < F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) >) dv_g \\ & -\int_M (|\nabla_A \Phi|^2 div X - \sum 2 < \nabla_{\nabla_{e_i} X} \Phi, \nabla_{e_i} \Phi >) dv_g \\ & -\frac{\lambda}{4}\int_M (|\Phi|^2 - m^2)^2 div X dv_g, \end{aligned} \tag{2.32}$$

where X is a variation vector field with compact support set and $\{e_i\}$ is an orthonormal frame of TM.

We may assume that S consists of a single point p. For $r > 0$ we take a cut-off function $\eta_r \in C_0^\infty(M)$ satisfying: $0 \leq \eta_r \leq 1$, $|\nabla \eta_r| \leq \frac{2}{r}$ in M and $\eta_r(x) = 1$, if $x \in B_p(r)$; $\eta_r(x) = 0$, if $x \in M \setminus B_p(2r)$. Since (A, Φ) is a Coupled Yang-Mills field on $M \setminus B_p(r)$ for any $r > 0$, we have $\Psi(X - \eta_r X) = 0$ for any $r > 0$. Thus, we have

$$\begin{aligned}|\Psi(X)| &= |\Psi(\eta_r X)| \leq C \int_M h^2(\eta_r|\nabla X| + |\nabla \eta_r||X|)\, dV_g \\ &\leq C(\int_{B_p(2r)} h^2|\nabla X|\, dV_g + \tfrac{1}{r}\int_{B_h 2r)} h^2|X|\, dV_g) \\ &\leq C(r^{n-4}\sup_M |\nabla X| + r^{n-5}\sup_M |X|)(\int_{B(P,2r)} h^{\frac{n}{2}}\, dV_g)^{\frac{4}{n}}.\end{aligned}$$

By conditions the right-hand side tends to 0 as $r \to 0$. Hence, we get $\Psi(X) = 0$ for any X. This shows that (A, Φ) is stationary on M.

□

3 Proof of theorem 1.1

Suppose that (A_i, Φ_i) be a sequence Coupled Yang-Mills fields on bundle E over Riemannian manifold (M, g). Let ϵ be sufficiently small and a be as in theorem 2.1. We define a closed subset for each i and $r > 0$ sufficiently small:

$$S_{i,r} = \{x \in M | r^{4-n} e^{ar^2} \int_{B_x(r)} h^2 dv_g \geq \epsilon\}. \tag{3.1}$$

It follows from the monotonicity formula (2.9) that $S_{i,r} \subset S_{i,r'}$ for any $r \leq r'$. By the standard diagonal process, we can choose a subsequence which also be denoted by $\{i\}$ such that for each k, the $S_{i,2^{-k}}$ converge to a closed subset $S_{2^{-k}}$. It is obvious that $S_{2^{-k}} \subset S_{2^{-l}}$ for $k \geq l$. Then, put

$$S = \cap_k S_{2^k}. \tag{3.2}$$

Discussing like in [Ti] (Proposition 3.1.2), one can prove that S is of Hausdorff codimension at least 4.

Assume that $x \in M \setminus S$, then by the definition of S, there exist r less than $dist(x, S)$ and $injrad(x)$, and $N = N(x) > 0$, such that for $i \geq N$, we have:

$$r^{4-n} \int_{B_x(r)} h^2(A_i, \Phi_i) dv_g < \epsilon. \tag{3.2}$$

Since the left-hand side of (3.2) is scaling invariant, we may rescale to assume that $r = 4$.

In what follows we shall only consider those $i \geq N$ when we restrict our attention on the geodesic ball $B_x(r)$. The point wise a priori estimate for Coupled Yang-Mills fields (theorem 2.2) and (3.2) imply that there exists a uniform constant $C_5 > 0$ such that

$$\sup_{B_x(1)} h^2(A_i, \Phi_i) \leq C_5 \epsilon, \tag{3.3}$$

if we assume that ϵ is less than that in theorem 2.2. Thus we have

$$\begin{aligned}&\sup_{B_x(1)} |F_{A_i}|^2 + |\nabla_{A_i} \Phi_i|^2 \leq C_5 \epsilon, \\ &\sup_{B_x(1)} |\Phi_i|^2 \leq C_6\end{aligned} \tag{3.4}$$

where C_6 is a uniform constant.

Fix a local trivialization for E on $B_x(1)$. If ϵ is sufficiently small, by the fist inequality in (3.4), we may apply the result of Uhlenbeck ([Uh1] theorem 2.7) to find Coulomb gauges for A_i on $B_x(1)$. In other words, there exist gauge transformations σ_i such that the connections $A'_i = \sigma_i(A_i)$ satisfy

$$d^* A'_i = 0, \quad on B_x(1), \tag{3.5}$$

$$d^*_\psi A'_{i,\psi} = 0, \quad on \partial B_x(1), \tag{3.6}$$

$$\|A'_i\|_{L^\infty(B_x(1))} \leq C_7 \|F_{A_i}\|_{L^\infty(B_x(1))} \leq C_8 \sqrt{\epsilon}, \tag{3.7}$$

where $A_{i,\psi}$ denote the restriction of A_i on $\partial B_x(1)$, and C_7, C_8 are uniform constants.

Let $\Phi'_i = \sigma_i \cdot \Phi_i$. The equations (2.3) and (3.4) are gauge invariant and hence they hold if we replace (A_i, Φ_i) by (A'_i, Φ'_i). We observe that (2.3) and (3.5) form an elliptic system for (A'_i, Φ'_i) over $B_x(1)$. By the standard elliptic theory, we conclude that the bounds on derivatives of (A'_i, Φ'_i) are uniform in i, hence by passing to a subsequence, $\sigma_i^*(A_i, \Phi_i)$ converges to a Coupled Yang-Mills field (A', Φ') in smooth topology on $B_x(\frac{1}{2})$.

We may cover the set $M \setminus S$ by a countable union of balls $B_{x_\alpha}(r_\alpha)$ such that (3.2) apply with $r = 8r_\alpha$ and $x = x_\alpha$. Applying the above analysis to each ball $B_{x_\alpha}(8r_\alpha)$, by passing to a subsequence, there exist smooth gauge transformations $\sigma_{i,\alpha}$, such that the sequence $\sigma_{i,\alpha}^*(A_i, \Phi_i)$ converges in smooth topology to a Coupled Yang-Mills field $(A'_\alpha, \Phi'_\alpha)$ on $B_{x_\alpha}(r_\alpha)$. We can now use a standard diagonal process of gluing gauges ([DK], Theorem 4.4.8), again passing to a subsequence, to obtain smooth gauge transformations σ_i on $M \setminus S$, such that $\sigma_i^*(A_i, \Phi_i)$ converge to a field (A, Φ) in smooth topology on compact subsets of $M \setminus S$. So, we have proved theorem 1.1.

4 Proof of theorem 1.2

Proposition 4.1 ([Si]): *Let (A, Φ) be a stationary Coupled Yang-Mills field on $M \setminus S$, where S is a discrete set. If $\int_K h(A, \Phi)^{\frac{n}{2}} dV_g < \infty$ for each compact set $K \subset M$, where $n > 4$ is the dimension of M, then there exist a gauge transformation σ such that $\sigma^*(A, \Phi)$ can be extended to be a smooth Coupled Yang-Mills field on M.*

Theorem 4.2: *Let $\{(A_i, \Phi_i)\}$ be a sequence of smooth Coupled Yang-Mills fields on E with $\int_M h^{\frac{n}{2}} dv_g \leq \Lambda$; then there exist a subsequence $\{\alpha\} \subset \{i\}$ and a (possibly empty) finite set $\tilde{S} = \{p_k\}_{k=1}^l$ of M satisfying the following:*

(1), the subsequence (A_α, Φ_α) converge to a smooth Yang-Mills connection (A, Φ) in the C^∞-topology on $M \setminus \Sigma$.

(2), for each $k = 1, \ldots, l$, there exists constants $\theta_k > 0$ such that

$$h(A_\alpha, \Phi_\alpha)^{\frac{n}{2}} dV_g \longrightarrow h(A, \Phi)^{\frac{n}{2}} dV_g + \sum_{k=1}^{l} \theta_k \cdot \delta_{p_k} \tag{4.1}$$

weakly in the sense of Radon measures on M. Here δ_{p_k} denotes dirac measure.

Proof: Let ϵ be as in the proof of theorem 1.1. We define a closed subset for each i and $r > 0$;

$$E_{i,r} = \{x \in M | \int_{B_r(x)} h(A_i, \Phi_i)^{\frac{n}{2}} dV_g \geq \epsilon\}. \tag{4.2}$$

It is obvious that $E_{i,r} \subset E_{i,R}$ for any $r \leq R$. By the standard diagonal process, we can choose a subsequence $\{i_j\}$ of $\{i\}$ such that for each k, the $E_{i_j, 2^{-k}}$ converge to a closed subset $E_{2^{-k}}$. Then $E_{2^{-k}} \subset E_{2^{-l}}$ for $k \geq l$.

Put $S = \cap_k E_{2^{-k}}$. We first claim that S is at most a finite set. We fixed an arbitrary compact set $K \subset M$. For any $\delta > 0$ sufficiently small, let $\{B_{x_b}(4\delta)$ be any finite covering of $S \cap K$ such that $x_b \in S \cap K$; $B_{x_b}(2\delta) \cap B_{x_c}(2\delta) = \emptyset$ for $b \neq c$. Take k big enough such that $2^{-k} < \delta$. Then for j sufficiently large, there are $y_b \in E_{i_j, 2^{-k}}$ such that $d(x_b, y_b) < \delta$. Then $\{B_{y_b}(5\delta)\}$ is a finite covering of $S \cap K$ and $B_{y_b}(\delta) \cap B_{y_c}(\delta) = \emptyset$ for $b \neq c$. On the other hand, for each b

$$\int_{B_{y_b}(\delta)} h(A_{i_j}, \Phi_{i_j})^{\frac{n}{2}} \, dV_g \geq \epsilon. \tag{4.3}$$

Summing up, we get

$$I \leq \frac{1}{\epsilon} \sum_{b=1}^{I} \int_{B_{y_b}(\delta)} h(A_{i_j}, \Phi_{i_j})^{\frac{n}{2}} \, dV_g \leq \frac{\Lambda}{\epsilon}, \tag{4.4}$$

This shows $\mathtt{H}^0(S \cap K) \leq \frac{\Lambda}{\epsilon}$ where $\mathtt{H}^0$ denotes the 0-dimensional Hausdorff measure on M. Since the 0-dimensional Hausdorff measure coincides with the counting measure, $S \cap K$ is at most finite. So S is at most a finite points set.

On the other hand, using corollary 2.3 and discussing like that in the proof of theorem 1.1, one can prove that there exists a subsequence $\{\tilde{i}\} \subset \{i\}$and gauge transformations $\sigma(\tilde{i})$, such that $\sigma(\tilde{i})^*(A_{\tilde{i}}, \Phi_{\tilde{i}})$ converge to a Coupled Yang-Mills field (A, Φ) in C^∞-topology on any compact subset outside S. Using Fatou's lemma, we have

$$\int_M h(A, \Phi)^{\frac{n}{2}} \, dV_g \leq \liminf_{\tilde{i} \to \infty} \int_M h(A_{\tilde{i}}, \Phi_{\tilde{i}})^{\frac{n}{2}} \, dV_g \leq \Lambda. \tag{4.5}$$

By Proposition 4.1, there exists a gauge transformation σ such that $\sigma^*(A, \Phi)$ extends to be a smooth Coupled Yang-Mills field on M.

Define

$$\tilde{S} = \bigcap_{r>0} \{x \in M | \liminf_{\tilde{i} \to \infty} \int_{B_(r)} h(A_{\tilde{i}}, \Phi_{\tilde{i}})^{\frac{n}{2}} \, dV_g \geq \epsilon\}. \tag{4.6}$$

Now we want to show that $\tilde{S}$ is contained in the above S. In fact, for any $x_0 \in M \setminus S$, if r is sufficiently small,

$$\int_{B_{x_0}(r)} h(A,\Phi)^{\frac{n}{2}}\, dV_g < \frac{\epsilon}{4}.$$

This implies that for $\tilde{i}$ sufficiently large,

$$\int_{B_{x_0}(r)} h(A_{\tilde{i}},\Phi_{\tilde{i}})^{\frac{n}{2}}\, dV_g < \frac{\epsilon}{2}.$$

Hence, $x_0 \in M \setminus \tilde{S}$. This shows that $\tilde{S} \subset S$.

Suppose $x_0 \in S \setminus \tilde{S}$; then there is an $r_0 > 0$ such that

$$\int_{B_{x_0}(r_0)} h(i')^{\frac{n}{2}}\, dV_g < \epsilon$$

for some subsequence $i' \to \infty$. By Corollary 2.3 ,

$$\sup_{x\in B(x_0,\frac{1}{2}r_0)} h(i') \le C_9 \cdot r_0^2 \cdot \epsilon^{\frac{2}{m}}$$

for some uniform constant C_9. This implies that there exists some subsequence of $\{A_{i'},\Phi_{i'}\}$ (modulo gauge transformations) converge in $B_{x_0}(\frac{1}{2}r_0)$ in the C^∞ topology. Then, there exists a subsequence $\{(A_\alpha,\Phi_\alpha)\}$ and a finite set $\tilde{S}$ such that $\{(A_\alpha,\Phi_\alpha)\}$(modulo gauge transformations) converges to a Coupled Yang-Mills field (A,Φ) in the C^∞ topology on $M \setminus \tilde{S}$.

Consider the Radon measure $\mu_\alpha = h(\alpha)^{\frac{n}{2}} dV_g$. By taking a subsequence if necessary, we may assume that $\mu_\alpha \to \mu$ weakly on M as Radon measures. Let us write (by Fatou's lemma)

$$\mu = h(A,\Phi)^{\frac{n}{2}} dV_g + \nu \tag{4.7}$$

for some nonnegative Radon measure ν on M. Since $\{(A_\alpha,\Phi_\alpha\}$ converges to (A,Φ) in the C^∞ topology on $M \setminus \tilde{S}$, the support of measure ν is contained in the discrete set $\tilde{S}$. Thus, we have $\nu = \sum_{k=1}^{l} \theta_k \delta_{p_k}$ for some $\theta_k \ge 0$ where we set $\tilde{S} = \{p_k\}_{k=1}^{l}$.

We show each θ_k is positive. Fix any p_k. For arbitrarily small $r > 0$, we take a cut-off function $\eta_r \in C_0^\infty(M)$ satisfying $0 \le \eta_r \le 1$ in M and $\eta_r(x) = 1$ if $x \in B_{p_k}(r)$; $\eta_r(x) = 0$ if $x \in M \setminus B_{p_k}(2r)$. By definition of $\tilde{S}$ we have

$$\epsilon \le \liminf_{\alpha\to\infty} \int_{B_{p_k}(r)} h(\alpha)^{\frac{n}{2}}\, dV_g \le \lim_{\alpha\to\infty} \int_M \eta_r h(\alpha)^{\frac{n}{2}}\, dV_g \le \theta_k + \int_{B_{p_k}(2r)} h(A,\Phi)^{\frac{n}{2}}\, dV_g. \tag{4.8}$$

Letting $r \to 0$, we obtain $\theta_j \ge \epsilon > 0$. This completes the proof.

□

In the following theorem, we will discuss the bubbling phenomenon of Coupled Yang-Mills fields.

Theorem 4.3: *Let $\{(A_\alpha, \Phi_\alpha)\}$,$\tilde{S}$ be as in Theorem 4.2; and $p \in \tilde{S}$. Then there are linear transformations $\tau_\alpha : T_pM \to T_pM$ such that a subsequence of $\tau_\alpha^* \exp_p^*(A_\alpha, \Phi_\alpha)$ converges to a smooth Coupled Yang-Mills field (B, Ψ) on a trivial bundle over (T_pM, g_p); and satisfying $h(B, \Psi) \neq 0$ and $\int_{T_pM} h(B, \Psi)^{\frac{n}{2}}\, dv_{g_p} \leq \theta_p$; where θ_p is determined in Theorem 4.2.*

Proof: We take a normal coordinate system $\{x^1, ..., x^n\}$ on the geodesic ball $B_p(2R)$ centered at p, and assume that $\tilde{S} \cap B_p(2R) = \{p\}$. Let $B(x, r)$ be the open ball in the normal coordinates with center x and radius r and let $B(r) = B(0, r)$. Defining the concentration function

$$Y_\alpha(t) = \sup_{y \in B(R)} \int_{\exp_p(B(y,t))} h(A_\alpha, \Phi_\alpha)^{\frac{n}{2}}\, dV_g, \tag{4.9}$$

for any $0 \leq t < R$. Each function Y_α is continuous and non-decreasing in t, and $Y_\alpha(0) = 0$. On the other hand, by the definition of $\tilde{S}$, we have

$$Y_\alpha(R) \geq \int_{B_p(R)} h(A_\alpha, \Phi_\alpha)^{\frac{n}{2}}\, dV_g \geq \frac{3\epsilon}{4} \tag{4.10}$$

holds for sufficiently large α. By continuity of Y_α, there exist $0 < r_\alpha < R$ and $x_\alpha \in \overline{B(R)}$ such that

$$Y_\alpha(r_\alpha) = \int_{\exp_p(B(x_\alpha, r_\alpha))} h(A_\alpha, \Phi_\alpha)^{\frac{n}{2}}\, dV_g = \frac{\epsilon}{2}. \tag{4.11}$$

Since the p is the a unique point in $\tilde{S} \cap B_p(2R)$, we obtain $r_\alpha \to 0$, $x_\alpha \to 0$, as $\alpha \to \infty$. Defining linear transformations $\tau_\alpha(x) = x_\alpha + r_\alpha \cdot x$ on T_pM. Let $U(\alpha) = B(-\frac{x_\alpha}{r_\alpha}, \frac{2R}{r_\alpha}) \subset T_PM$. It is easy to see that $B(2R) = \sigma_\alpha(U(\alpha))$. Since x_α lies in $B(\frac{R}{2})$ for sufficiently large α, we have $B(\frac{R}{r_\alpha}) \subset U(\alpha)$, which leads to $U(\alpha) \to T_PM$ as $\alpha \to \infty$.

We set $B_\alpha = \tau_\alpha^* \exp_p^*(A_\alpha)$, $\Psi_\alpha = \tau_\alpha^* \exp_p^*(\Phi_\alpha)$. We can easily see (B_α, Ψ_α) are Coupled Yang-Mills fields on pull backed bundles over $(U(\alpha), g_\alpha)$, where the metric $g_\alpha = r_\alpha^{-2} \tau_\alpha^* \exp_p^* g$. Note that the based manifolds (T_pM, g_α) converge to $(T_pM, g_p) \cong R^n$ as $\alpha \to \infty$. By the definition of (B_α, Ψ_α), x_α, r_α, we have

$$\int_{U(\alpha)} h(B_\alpha, \Psi_\alpha)^{\frac{n}{2}}\, dV_{g_\alpha} = \int_{B_p(2R)} h(A_\alpha, \Phi_\alpha)^{\frac{n}{2}}\, dV_g \leq \Lambda, \tag{4.12}$$

and

$$Y_\alpha(r_\alpha) = \int_{B(1)} h(B_\alpha, \Psi_\alpha)^{\frac{n}{2}}\, dV_{g_\alpha} = \sup_{z \in \tau_\alpha^{-1}(B(R))} \int_{B(z,1)} h(B_\alpha, \Psi_\alpha)^{\frac{n}{2}}\, dV_{g_\alpha} = \frac{\epsilon}{2}. \tag{4.13}$$

Since $g_\alpha \to g_p$ in C^∞ topology as $\alpha \to \infty$, and the constants ϵ and C_5 in Corollary 2.3 depend only on λ and the bound of sectional curvature of metrics. If we assume that the ϵ in (4.13) is small enough, then we can apply corollary 2.3 for any α. So, we have

$$\sup_{B(z, \frac{1}{2})} h(B_\alpha, \Psi_\alpha) \leq C_{10} \epsilon^{\frac{2}{n}}$$

for any $z \in K$, here C_{10} is a uniform constant and K is an arbitrary compact set in T_pM. This implies that there exists a subsequence (B_β, Ψ_β) (modulo gauge transformations) converge to a smooth Coupled Yang-Mills field (B, Ψ) over $(T_p(M), g_p)$. Passing to the limit in (4.13), we have

$$\int_{B(1)} h(B, \Psi)^{\frac{n}{2}} \, dx = \frac{\epsilon}{2}. \tag{4.14}$$

This shows that $h(B, \Psi) \neq 0$. By Fatou's lemma, we have

$$\begin{aligned} \textstyle\int_{T_pM} h(B, \Psi)^{\frac{n}{2}} \, dx \quad & \leq \liminf_{\beta \to \infty} \textstyle\int_{U(\beta)} h(B_\beta, \Psi_\beta)^{\frac{n}{2}} \, dV_{g_\beta}, \\ & \leq \theta_p + \textstyle\int_{B_p(2R)} h(A, \Phi)^{\frac{n}{2}} \, dV_g \end{aligned}$$

Letting $R \to 0$, we have

$$\int_{T_pM} h(B, \Psi)^{\frac{n}{2}} \, dx \leq \theta_p. \tag{4.15}$$

This completes the proof.

□

If the blow up set $\tilde{S}$, is not empty, from theorem 4.3, we know that there exists a bubbling Coupled Yang-Mills field (B, Ψ) over (R^n, g_0) satisfying $h(B, \Psi) \neq 0$ and

$$\int_{R^n} h(B, \Psi)^{\frac{n}{2}} \, dx \leq \infty.$$

But, this is contradictory with Corollary 2.4. So, the blow up set $\tilde{S}$ which we constructed in Theorem 4.2 must be empty. Then, the subsequence (A_α, Φ_α) (modulo gauge transformations) converges to a smooth Coupled Yang-Mills field (A, Φ) A in the C^∞-topology on M. This completes the proof of Theorem 1.2.

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